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The Phase Diagram of the Abelian Lattice Higgs Model. A Review of Rigorous Results

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We give a review (and some improvements) of the rigorous results on the phase structure of Abelian lattice Higgs models with gauge groups $U(1)$ and $\mathbb{R}$. Emphasis is put on the relation between the Higgs mechanism (gauge-independent) and spontaneous symmetry breaking (gauge-dependent). We also discuss some nonperturbative effects due to Gribov copies in this context.

KEY WORDS: Abelian lattice Higgs model; Higgs mechanism; spontaneous symmetry breaking; gauge fixing; Gribov copies.

1. INTRODUCTION

Scalar lattice QED comes basically in two versions: noncompact models with gauge group $\mathbb{R}$ and compact models with gauge group $U(1)$. Although it is hoped that both versions should have the same continuum limits, they have rather different phase structures, which coincide at most in the weak coupling regime.

In space-time dimension $d \geqslant 3$ the noncompact model is known to have two phases. In the Coulomb phase the photon is massless,[1-3] while in the Higgs phase it acquires a mass.[4] The standard explanation (see, e.g., Ref. 5) of this mass generation is based on the assumption that the gauge symmetry is broken spontaneously and the vacuum expectation value $\langle \phi \rangle$ of the Higgs field is nonzero.

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It seems that this argument is a persistent source of confusion, since Elitzur's theorem[6] states that *local* gauge symmetries cannot be broken *spontaneously*. The point is that continuum gauge theories always work with gauge-fixing terms, which *explicitely* break the local gauge invariance, and hence Elitzur's theorem does not apply. If, however, the gauge-fixing term still is invariant under *global* gauge transformations, then it is sensible to ask under what conditions this global symmetry is broken spontaneously, to what extent this may depend on the gauge fixing, and whether this has anything to do with the Higgs mechanism (defined by the existence of a mass gap, say).

The first rigorous result in this direction has been obtained by Kennedy and King,[7] who considered the noncompact model with (the lattice version of), the gauge-fixing term $(2\alpha)^{-1}(\partial\mu A^{\mu})^2$, where A^{μ} is the gauge potential. For these α-gauges, as they will be called from now on, they showed that in $d \leqslant 4$ the expectation value $\langle \phi \rangle_{\alpha}$ is zero for all $\alpha > 0$ independently of the parameters in the Higgs potential and the electromagnetic coupling constant e. In Landau gauge, however (which corresponds to the limit $\alpha \downarrow 0$), they showed (see also Ref. 8) that $\langle \phi \rangle_{\text{Landau}} \neq 0$ in a region of parameters overlapping with the Higgs phase, while it is zero in a region overlapping with the Coulomb phase.

Thus, for the *noncompact* model the semiclassical picture seems to be correct in Landau gauge, but misleading in all other α-gauges. Perturbatively this could have been anticipated from a loop expansion of the generating functional $G(J)$ of connected Green's functions (in the Higgs fields).[3] The derivative $\partial G/\partial J$ is nothing but the one-point function $\langle \phi \rangle$. On the one-loop level we get (among other terms) a tadpole contribution from the Goldstone propagator $(k^2)^{-1} + \alpha e^2 v^2 (k^2)^{-2}$, which is infrared-divergent in $d \leqslant 4$ unless α is zero ($|\phi| = v$ is the classical value, i.e., the minimum of the Higgs potential).

Hence, the tree approximation $|\langle \phi \rangle| = v$ is unrealiable for $\alpha > 0$, since the radiative corrections are infinite. This situation is very analogous to two-dimensional spin systems with continuous symmetries. There Goldstone tadpole diagrams as in Fig. 1 are always infrared-divergent and hence give a sign for the well-known Mermin–Wagner theorem[10,11] on the absence of spontaneous symmetry breaking (SSB) in these models.

For the compact $U(1)$ Higgs model the situation is by far not as clear. We first note that this model is perfectly well defined without any gauge fixing, and gauge-fixed versions are usually not investigated. But then $\langle \phi \rangle$

[3] In the language of statistical mechanics, $G(J)$ is the Gibbs free energy density [also called "pressure" by $P(\phi)_2$-theorists] as a function of an external "magnetic" field J coupled to the scalar field ϕ.

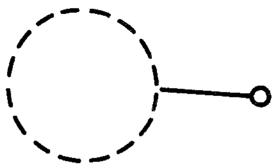

Fig. 1. Goldstone tadpole contribution to the one-point function. Propagators and vertices for the Abelian Higgs model can be found in, e.g., Ref. 9.

vanishes identically due to Elitzur's theorem. Hence, to compare with the noncompact model or with the continuum theory, it is interesting to consider α-gauges as well. Not too surprisingly it also turns out that the global gauge symmetry is never broken, provided $\alpha > 0$ and $d \leqslant 4$.[12,13] Moreover, we now have to deal with the fact that α-gauges in compact Abelian models still suffer from the existence of Gribov copies,[12 15] and hence the methods of Kennedy and King to show SSB in Landau gauge do not apply. As a special feature, due to the Abelian nature of the model, these Gribov copies are related by a subgroup of gauge transformations, which we call $\underline{\text{Grib}}$. If $\underline{\text{Grib}}$ is unbroken, then even the two-point function $\langle \bar{\phi}_x \phi_y \rangle$ vanishes at noncoinciding points $x \neq y$.

We would like to point out here that the presence of Gribov copies does not invalidate the Faddeev–Popov procedure as long as gauge-invariant expectations are considered.[15] This is also true in perturbation theory if one expands about the trivial minimum of the classical action and forgets about the Gribov copies.[14] But, as has also been argued in Ref. 14, the same perturbative procedure is bound to yield incorrect answers concerning gauge-dependent Green's functions. Indeed, we will report in this review on some new results of Ref. 13, where it is shown that in the compact model even in Landau gauge $\langle \phi(x) \rangle$ as well as $\langle \bar{\phi}(x)\, \phi(y) \rangle$ (for $x \neq y$) are zero in the strong coupling regime of the confinement/Higgs phase. This extends earlier results of Ref. 12 and implies that either $\langle \phi \rangle_{\text{Landau}}$ is not a good order parameter or it indicates a phase transition where there is none is terms of local gauge-invariant observables.

We conclude by mentioning some important contributions to the subject that had to be left out because of considerations of space. First, we apologize for not having given as much appreciation as they deserve to all the Monte Carlo results on these models. For an excellent review we refer the reader to Ref. 16. Second, we have not discussed issues and order parameters concerning the existence of charged states,[17,18] the particle structure,[19] and the Meissner effect,[13] all of which seem to be very physical and useful characterizations of the phase structure in lattice field theory.

The organization of this paper is as follows: In Section 2 we present what is known rigorously about the phase diagram of the noncompact Abelian lattice Higgs model. We show that the gauge-invariant description (i.e., via the photon mass) agrees very well with the appearence of SSB in Landau gauge. Our theorems are slight improvements of those present in the literature. The proofs are deferred to Section 4. Section 3 is devoted to the compact Abelian model. We first describe the phase diagram in terms of gauge-invariant observables by referring to existing analytical and numerical results. Then we summarize what is known about SSB in the gauge-fixed model and discuss possible answers to open questions. Finally, some technical details concerning Section 4 are given in the appendices.

2. THE NONCOMPACT MODEL

In this section we consider the Abelian Higgs model with gauge group $\mathbb{R}$ ("the noncompact $U(1)$"). Our lattice is a finite, open subcomplex[20,8] of $\mathbb{Z}^d$, i.e., consists of a finite set Λ_0 of points in $\mathbb{Z}^d$, the set of nearest neighbor pairs $\langle xy \rangle$ (also called links or bonds) in $\mathbb{Z}^d$, such that at least x or y is in Λ_0, the set Λ_2 of plaquettes that contain at least one link in Λ_1, etc. The gauge field is a real-valued 1-form on Λ, i.e., a function $\langle xy \rangle \mapsto A_{xy}$ living on links $\langle xy \rangle \in \Lambda_1$, with $A_{xy} = -A_{yx}$. The Higgs field ϕ_x lives on points in Λ_0 and takes values in the complex numbers $\mathbb{C}$. The Higgs field ϕ_x is often written as $R_x e^{i\varphi_x}$, with $\varphi_x \in [-\pi, \pi)$ and $0 \leqslant R_x < \infty$. An observable in Λ is a complex-valued function of the fields A_{xy}, resp. ϕ_x, on points, resp. links, in Λ.

Gauge transformations are given by

$$\phi_x \to \phi_x^\lambda = e^{i\lambda_x} \phi_x$$

$$A_{xy} \to A_{xy}^\lambda = A_{xy} + (d\lambda)_{xy}$$

where d denotes the exterior derivative on Λ, $(d\lambda)_{xy} = \lambda_y - \lambda_x$, and the gauge-transformed observable is defined by

$$f^\lambda(A^\lambda, \phi^\lambda) = f(A, \phi)$$

The action $S_\Lambda = S_\Lambda^{\text{inv}} + S_\Lambda^\alpha$ consists of a gauge-invariant part

$$S_\Lambda^{\text{inv}} = \frac{1}{2} \sum_{\langle xy \rangle \in \Lambda_1} |(D_A \phi)_{xy}|^2 + \frac{1}{2e^2} \sum_{p \in \Lambda_2} F(p)^2 + \sum_{x \in \bar{\Lambda}_0} V(|\phi_x|)$$

and the gauge-fixing term

$$S_\Lambda^\alpha = \frac{1}{2\alpha e^2} \sum_{x \in \Lambda^0} (d^*A)^2(x)$$

Here and in the following, sums (and products) over $\langle xy \rangle \in \Lambda_1$, resp. $p \in \Lambda_2$, are always meant as sums (and products) over positive-oriented links, resp. plaquettes, unless noted otherwise. $\bar{\Lambda}_0$ is the set of all points that are attached to at least one bond in Λ_1, $F = dA$ is the electromagnetic field strength, d^*A is the lattice divergence of A, and $D_A\phi$ is the covariant derivative of ϕ; V is the Higgs potential:

$$F(p) = (dA)(p) = \sum_{b \in \partial p} A_b$$

$$(d^*A)(x) = \sum_{y:|x-y|=1} A_{yx}$$

$$(D_A\phi)_{xy} = \phi_x - [\exp(-iA_{xy})]\phi_y$$

$$V(|\phi|) = \lambda(|\phi|^2 - v^2)^2$$

e^2, λ, and α are to be taken nonnegative and v^2 may be negative or positive.

We fix our boundary conditions by putting A_b and φ_x to zero whenever $b \notin \Lambda_1$ and $x \notin \Lambda_0$, respectively. The expectation value of a local observable f [i.e., a function $f(A, \phi)$ depending only on finitely many variables A_{xy} and ϕ_x] is given by

$$\langle f \rangle_\alpha = \lim_{\Lambda \to \mathbb{Z}^d} \langle f \rangle_{\Lambda,\alpha}$$

$$\langle f \rangle_{\Lambda,\alpha} = Z_{\Lambda,\alpha}^{-1} \int D_\Lambda A \, D_\Lambda \phi \, f \exp(-S_\Lambda^{\text{inv}}) \exp(-S_\Lambda^\alpha)$$

The normalization $Z_{\Lambda,\alpha}$ is chosen in such a way that $\langle 1 \rangle_{\Lambda,\alpha} = 1$ and

$$D_\Lambda A = \prod_{b \in \Lambda_1} dA_b, \qquad D_\Lambda \phi = \prod_{x \in \bar{\Lambda}_0} R_x \, dR_x \prod_{x \in \Lambda_0} d\varphi_x$$

As a convenient limit of the model just defined, one often considers the "fixed-length" version, which is obtained by sending $\lambda \to \infty$ and keeping $v^2 > 0$ fixed. In this limit the Higgs field becomes $\phi_x = ve^{i\varphi_x}$ and the gauge-invariant part of the action is

$$\tilde{S}_\Lambda^{\text{inv}} = -v^2 \sum_{\langle xy \rangle \in \Lambda_1} \cos(A - d\varphi)_{xy} + \frac{1}{2e^2} \sum_{p \in \Lambda_2} F(p)^2$$

In the following, Δ_Λ will denote the lattice Laplacian in the volume Λ, with Dirichlet boundary conditions, and for p-forms f and g, $(f, g)_\Lambda$ stands for the scalar product

$$(f, g)_\Lambda = \sum_{c \in \Lambda_p} \overline{f(c)}\, g(c)$$

Lemma 2.1. (i) Let f be an observable in Λ transforming under gauge transformations according to the irreducible representation q, i.e., $f^\lambda = \{\exp[i(q, \lambda)_\Lambda]\} f$ with some function q: $\Lambda_0 \to \mathbb{R}$. Then

$$\langle f \rangle_{\Lambda,\alpha} = \{\exp[-\tfrac{1}{2}\alpha e^2 (q, \Delta_\Lambda^{-2} q)_\Lambda]\} \langle f \rangle_{\Lambda,\alpha=0}$$

In particular, $\langle f \rangle_{\Lambda,\alpha}$ is independent of α for gauge-invariant observables.

(ii) In dimension $d \leqslant 4$

$$\langle \phi_x \rangle_\alpha = 0, \qquad \alpha > 0$$

independently of e^2 and the parameters of the Higgs potential.

Remark. The statement (ii) was first proven by Kennedy and King.[7] The following proof, however, which is taken from Ref. 3, displays much more explicitly the α-dependent spin wave contribution destroying long-range order in $d \leqslant 4$.

Proof of Lemma 2.1. (i) We consider the unnormalized expectation value

$$Z_{\Lambda,\alpha} \langle f \rangle_{\Lambda,\alpha} = \int D_\Lambda A\, D_\Lambda \phi\, f(A, \phi) \times \exp[-S_\Lambda^{\text{inv}}(A, \phi)] \exp[-S_\Lambda^\alpha(d^* A)]$$

and insert 1 in the form

$$1 = \det \Delta_\Lambda \int \delta(\Delta_\Lambda \lambda + d^* A)\, D_\Lambda \lambda$$

Since S_Λ^{inv} is gauge-invariant, we obtain, after a change of variables,

$$A \to A' = A^\lambda = A + d\lambda, \qquad \phi \to \phi' = \phi^\lambda = e^{i\lambda} \varphi$$

$$Z_{\Lambda,\alpha} \langle f \rangle_{\Lambda,\alpha} = \det \Delta_\Lambda \int D_\Lambda \lambda\, D_\Lambda A'\, D_\Lambda \phi'\, \delta(d^* A') \times f(A') \exp[i(q, \lambda)_\Lambda] \exp[-S^{\text{inv}}(A', \phi')] \times \exp[-S_\Lambda^\alpha(d^* A' - \Delta_\Lambda \lambda)]$$

where we have used the transformation properties of f and the fact that $d^*d\lambda = \Delta_\Lambda \lambda$. Due to the δ-functions, we can drop d^*A' in S^α_Λ. But then the integral over λ factorizes from the rest and we obtain

$$\langle f \rangle_{\Lambda,\alpha} = \langle f \rangle_{\Lambda,\alpha=0} Z^{-1}_{\text{g.f.}} \int D_\Lambda \lambda \exp[-S^\Lambda_\alpha(-\Delta_\Lambda \lambda)] \exp[i(q, \lambda)_\Lambda] \tag{2.11}$$

with

$$Z_{\text{g.f.}} = \int D_\Lambda \lambda \exp[-S^\Lambda_\alpha(-\Delta_\Lambda \lambda)]$$

The integral in (2.11) is Gaussian and can be evaluated, giving Lemma 2.1(i).

(ii) Due to Lemma 2.1(i),

$$\langle \phi_x \rangle_{\Lambda,\alpha} = \{\exp[-\tfrac{1}{2}\alpha e^2(\Delta_\Lambda^{-2})_{xx}]\} \langle \phi_x \rangle_{\Lambda,\alpha=0}$$

Since $(\Delta^{-2})_{xx}$ is infrared-divergent in $d \leqslant 4$ and $\langle \phi_x \rangle_{\alpha=0}$ is finite due to the $\lambda\phi^4$ term in the action, part (ii) is proven. ∎

We now summarize what is known about the phase diagram of the noncompact model. We say that the "photon is massive" if there is a constant $m > 0$ such that

$$|\langle F(p)\, F(p') \rangle| \leqslant \text{const} \cdot e^{-md(p,p')}$$

where $d(p, p')$ is the distance between p and p' and the constants const and m are independent of the choice of p and p'. We say that the decay of the photon propagator is summable if for any two plaquettes p and p'

$$\sum_{x \in \mathbb{Z}^d} |\langle F(p_x)\, F(p') \rangle| < \infty$$

Here p_x is the translate of p by x. Note that a massive photon implies summable decay of the photon propagator.

We define the Higgs phase to be that region in the parameter space $\{(e^2, v^2, \lambda)\}$ where the photon is massive, and the Coulomb phase to be the region where the photon propagator is not summable (it is generally believed that there is no phase with massless but still summable photon propagator). Based on the results to be presented below, one qualitatively expects a phase diagram as in Fig. 2.

We first focus on the fixed-length model ($\lambda = \infty$) and state a condensed set of results in Theorem 2.2 and Corollary 2.3. The statements are visualized in Fig. 3.

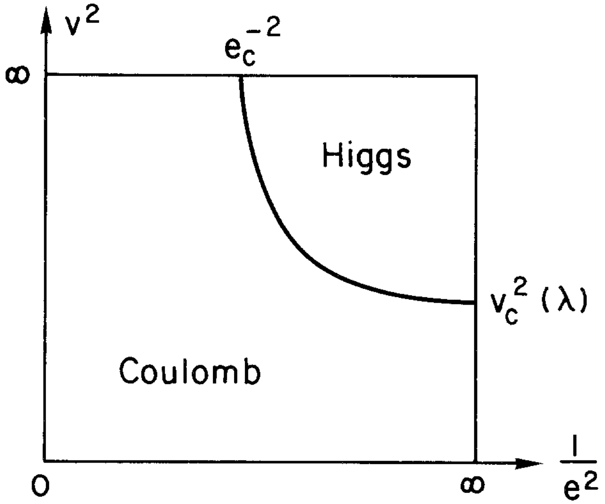

Fig. 2. Expected phase diagram in the noncompact model for arbitrary but fixed $\lambda > 0$. We conjecture $\langle \phi \rangle_{\text{Landau}} = 0$ in the Coulomb phase and $\langle \phi \rangle_{\text{Landau}} \neq 0$ in the Higgs phase; $v_c^2(\lambda)$ is the critical point in the pure $O(2)$ symmetric spin model, which is obtained in the limit $e^2 = 0$.

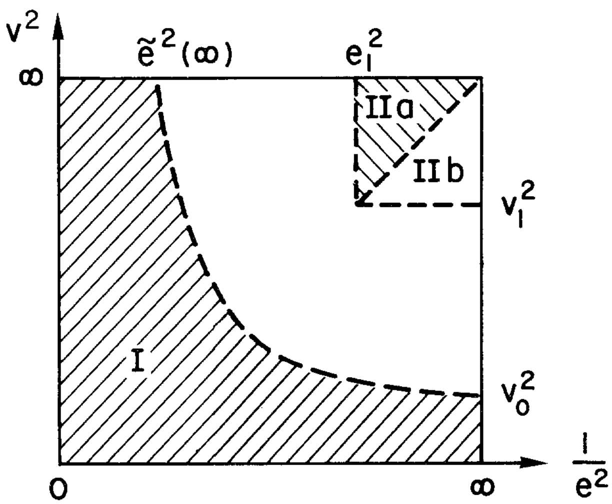

Fig. 3. Established phase diagram of the noncompact fixed-length model. The Coulomb phase and $\langle \phi \rangle_{\text{Landau}} = 0$ are proven in region I. The Higgs phase is proven in IIa. $\langle \phi \rangle_{\text{Landau}} \neq 0$ is known in IIa and IIb.

Theorem 2.2. Consider the noncompact, Abelian, fixed-length model in $d \geqslant 3$.

(i) There is a constant $v_0^2 > 0$ and a bounded function $\tilde{e}^2(\cdot)$, strictly monotone increasing with $\tilde{e}^2(v_0^2) = 0$, such that for $e^2 > \tilde{e}^2(v^2)$ the photon propagator is not summable and $\langle \phi \rangle_{\text{Landau}} = 0$.

(ii) There are constants $e_1^2 > 0$ and $v_1^2 > 0$ such that for $e^2 < e_1^2$ and $e^2 v^2 > e_1^2 v_1^2$ the photon is massive and $\langle \phi \rangle_{\text{Landau}} > 0$.

Corollary 2.3.[7] If $v^2 > v_1^2$, then $\langle \phi \rangle_{\text{Landau}} > 0$ for all $e^2 < e_1^2$.

Proof of Corollary 2.3. By a correlation inequality,[21] $\langle \phi \rangle_{\text{Landau}}$ is a decreasing function of e^2. Corollary 2.3 follows immediately from Theorem 2.2(ii). ∎

Remarks.

1. Theorem 2.2(i) improves earlier results of Ref. 1–3 and will be proven in Section 4.

2. The existence of a massive Higgs phase in this model was announced in Ref. 1 and "quasiproven" in Quasi-Theorem 3.20 of Ref. 22. Here it is a special case of Theorem 2.5(i) below, which is proven in Section 4.

3. For the proof of SSB in Landau gauge, we refer to Kennedy and King[7] (see also Ref. 8).

Next we turn to the full model with arbitrary finite values of λ. In this case the proofs of what is known are either more involved or the results not quite as strong. In any case, they have to be differentiated a little bit more. We first state the analogue of Theorem 2.2(i), for which we only know the following weaker version.

Theorem 2.4. Let $\lambda > 0$ and $d \geqslant 3$ in the noncompact model.

(i) The photon propagator is not summable, provided $e^2 > \tilde{e}^2(\infty)$,[3] or $v^2 < v_0^2$ and $\lambda > \lambda_0$ for some positive constants v_0 and λ_0.[2] If v^2 is negative, the same holds for arbitrary nonnegative values of λ and e^2 (see Section 4).

(ii) $\langle \phi \rangle_{\text{Landau}} = 0$, provided $e^2 > \tilde{e}^2(\infty)$,[3] or $v^2 < v_c^2(\lambda)$, where $v_c^2(\lambda) > 0$ is the critical value for SSB in the pure spin model (Ref. 7, same proof as for Corollary 2.3).

Remark. One of course expects that Theorem 2.4(i) holds with v_0^2 replaced by $v_c^2(\lambda)$, but there is no proof so far.

Now we come to the generalizations of Theorem 2.2(ii) for finite values of λ. There are two kinds of cluster expansions available in this case.

For large λ one can use the expansion of Kennedy and King[7] to prove a massive Higgs phase (see Section 4). For small λ one expands about the massive Gaussian approximation of perturbation theory.[4] Finally, $\langle \phi \rangle_{\text{Landau}}$ can be bounded below by its value in the fixed-length model with a shifted parameter v due to a correlation inequality of Ref. 7. Accordingly, we have the following result.

Theorem 2.5. Let $\lambda > 0$ and $d \geqslant 3$ in the noncompact model.

(i) The photon is massive, provided $\lambda > \lambda_1$, $e^2 < e_1^2$, and $ev > e_1 v_1$ for suitable positive constants λ_1, v_1, and e_1 (for proof see Section 4).

(ii) The photon is massive for any positive values of $\sigma := \lambda v^2$ and $\mu = ev$, provided λ is small enough, i.e., $\lambda < \lambda_0(\sigma, \mu)$.[4]

(iii) $\langle \phi \rangle_{\text{Landau}} > 0$ provided $e^2 < e_1^2$ and $v^2 > 4v_1^2 + d/2\lambda$.[7]

Theorems 2.4 and 2.5 are summarized in Fig. 4.

In conclusion, we feel that the phase diagram of the Abelian lattice Higgs model with gauge group $\mathbb{R}$ is quite well understood. There is a Higgs phase with massive photons and a Coulomb phase with massless photons. The vacuum expectation value of the Higgs field in Landau gauge seems indeed to be a good order parameter for this phase transition, while $\langle \phi \rangle_\alpha = 0$ in all other α-gauges and in both phases.

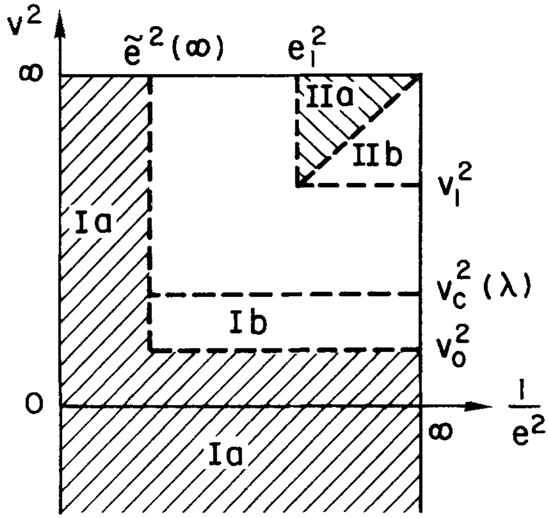

Fig. 4. Established phase diagram of the noncompact model for large but fixed values of λ. The Coulomb phase is proven in region Ia, the Higgs phase in region IIa, and $\langle \phi \rangle_{\text{Landau}} \neq 0$ is known in IIa + IIb. For small λ the Coulomb phase is only proven with $v_0^2 = 0$ and the precise shape of region IIa is not known, unless we know the bound $\lambda_0(\sigma, \mu)$ in Theorem 2.7(ii) explicitely. This is not worked out in Ref. 4, but certainly could be done with some more work.

3. THE COMPACT MODEL

Here the gauge group $\mathbb{R}$ is replaced by $U(1)$, i.e., the Higgs field still takes values in $\mathbb{C}$, but the gauge field A_{xy} is an angle $A_{xy} \in [-\pi, \pi)$. In other words, the basic variables are now the so-called parallel transporters $U_{xy} = \exp(iA_{xy})$. Consequently, the terms $F(p)^2$ and $(d^*A)^2(x)$ are replaced by $2\cos F(p)$ and $2\cos(d^*A)$, respectively. We have

$$S_A^{\text{inv}} = \frac{1}{e^2} \sum_{p \in \Lambda_2} \cos F(p)$$

$$+ \frac{1}{2} \sum_{\langle xy \rangle \in \Lambda_1} |(D_A \phi)_{xy}|^2 + \sum_{x \in \bar{\Lambda}_0} V(|\phi_x|) \tag{3.1a}$$

$$S_A^{\alpha} = \frac{1}{\alpha e^2} \sum_{x \in \Lambda_0} \cos(d^*A)(x) \tag{3.1b}$$

With the same boundary conditions as in Section 2 it is easy to see that again $\langle f \rangle_{\Lambda,\alpha}$ is independent of α for gauge-invariant observables in Λ.

Let us recall, however, that now, since the range of integration for the gauge field A is compact, unnormalized expectations are well defined even in the limit $\alpha^{-1} = 0$, i.e., without any gauge fixing.

Compact (Abelian and non-Abelian) Higgs models without gauge fixing were studied analytically in the pioneering work of Osterwalder and Seiler.[(23)] The surprising result was that for λ large enough, the Higgs phase extends all the way down to $v^2 = -\infty$, provided e^2 is large enough. In this limit the radial degree of freedom is frozen to zero and we obtain the confinement phase of the $U(1)$ pure gauge model.[4]

The above result implies that in the theory without gauge fixing, expectations of *local* observables depend analytically on λ, e^2, and v^2 in a corresponding neighborhood. Hence in this region there is no *local, gauge-invariant* order parameter that could distinguish between a "Higgs subregion" and a "confinement subregion." Consequently, one speaks of the Higgs/confinement phase.

For small λ, however, Monte Carlo data indicate that the situation may very well change (for a very nice review and a rather complete list of references to all relevant Monte Carlo results, see Jersák[(16)]). Indeed, the only rigorous result in this case can be taken from Balaban *et al.*,[(4)] and the statement is precisely the same as for the noncompact model [see Theorem 2.5(ii)]. In particular, their cluster expansion breaks down in the nonperturbative region, where e^2 is large or v^2 small, which is in

[4] For $\lambda = \infty$ (i.e., in the fixed-length model) only $v^2 \geq 0$ has to be considered (by shifting $A_{xy} \to A_{xy} + \pi$). The pure $U(1)$ gauge model is then obtained for $v^2 = 0$.

accordance with the numerical evidence for a phase separation between the confinement and the Higgs phase at small λ. Summarizing the above remarks, we have the following result.

Theorem 3.1. Let $d \geq 2$ in the compact model.

(i) There is a constant $\lambda_0 < \infty$ such that for all $\lambda > \lambda_0$ the photon is massive, provided $e^2 > \tilde{e}_\lambda^2(v^2)$ for some bounded, positive function $\tilde{e}_\lambda^2(\cdot)$ with $\tilde{e}_\lambda^2(\infty) = 0$.[23]

(ii) The photon is massive for any positive values of $\sigma := \lambda v^2$ and $\mu := ev$ provided λ is small enough, i.e., $\lambda > \lambda_0(\sigma, \mu)$.[(4)]

The question of whether or in what sense there is also a massless Coulombic phase in this model remains unsettled (except for the limit of the pure $U(1)$ gauge model, which in $d=4$ is deconfining[20,24] and massless[25] for small values of e^2). However, there have been very interesting proposals and investigations of (ratios of) string observables $\phi_x U(C_{xy}) \phi_y$ [$U(C_{xy})$ is a product of link variables along some path C_{xy} from x to y], which are related to the existence of charged states (respectively bound states) between a dynamical and an external charge ("mesons") and give indeed a hint for a separate "non-Higgs" (Coulombic?) phase in this model in $d=4$ (see Fig. 5).[17,19]

We now come to the question of SSB in the gauge-fixed model and try to see how the results may fit into the above gauge-invariant description of

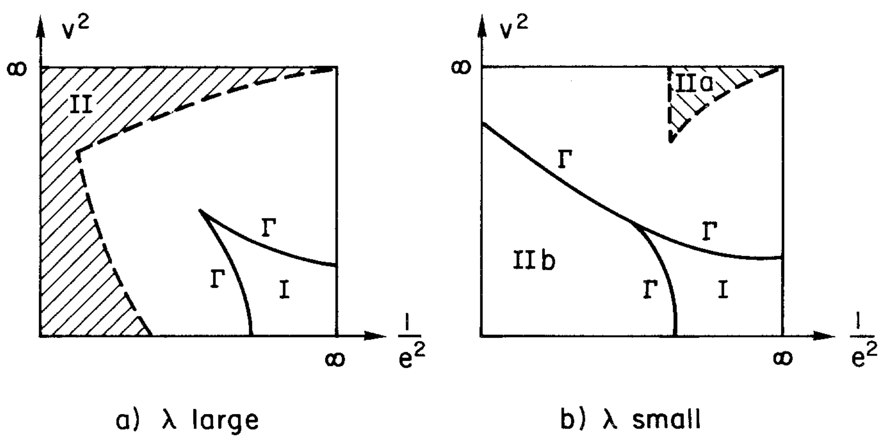

Fig. 5. Phase diagram of the compact model in $d=4$. (a) For large λ the Higgs/confinement phase is proven in region II. Region I is suggested to be massless and the existence of a phase separation line Γ is widely accepted by those working with Monte Carlo approaches. Only the endpoints, however, are rigorously established. (b) For small λ the Higgs phase is only proven in region IIa. Monte Carlo data indicate a separate confinement phase IIb.

the phase diagram. We first note that whereas the action $S_\Lambda = S_\Lambda^{\text{inv}} + S_\alpha$ had a unique minimum ($A = 0$, $\phi = v$) in the noncompact case (the remaining global symmetry was fixed by our boundary conditions), this is not so in the compact one. In fact, it can be seen that these "Gribov copies" are related by a certain symmetry of the action of our model: S_Λ is still invariant under the following subgroup of gauge transformations

$$\underline{\text{Grib}}_\Lambda := \{g(\cdot) = e^{i\lambda(\cdot)}\colon e^{i(\Delta\lambda)(x)} = 1 \ \forall x \in \Lambda_0\}$$

Note that even though $\Delta_\Lambda \lambda = 0$ implies $\lambda = 0$ (we have chosen Dirichlet b.c.), $e^{i\Delta_\Lambda\lambda} = 1$ does not imply $e^{i\lambda} = 1$. Indeed, the number of elements of $\underline{\text{Grib}}_\Lambda$ is exactly given by $\det \Delta_\Lambda$.[13,14] In the infinite volume, $\underline{\text{Grib}}$ is even continuous and contains the global (i.e., x-independent) gauge transformations $\underline{\text{Glob}}$ as a trivial subgroup.

An important consequence of this symmetry is the fact that not only $\langle \phi(x) \rangle_\alpha$, but also the two-point function $\langle \bar\phi(x)\, \phi(y) \rangle_\alpha$ is zero for $x \neq y$ if $\underline{\text{Grib}}$ is not broken. Indeed, by invariance under $\underline{\text{Grib}}$ we have for any $g = e^{i\lambda} \in \underline{\text{Grib}}$

$$(\bar\phi_x \phi_y \rangle_\alpha = e^{i(\lambda_x - \lambda_y)} \langle \bar\phi_x \phi_y \rangle_\alpha$$

Hence, the two-point function vanishes at noncoinciding points if we can find for any $x \neq y$ an element $g = e^{i\lambda} \in \underline{\text{Grib}}$ such that $g(x) \neq g(y)$. This can be achived by choosing $\lambda(x) = ax$ with suitable $a \in \mathbb{R}^d$.

Let us first concentrate on the symmetry $\underline{\text{Glob}}$ and state the analogues of Lemma 2.1 and Theorem 2.4(ii) for the compact model:

Theorem 3.2.[12,13] Consider the compact model with $\lambda > 0$.

(i) For $\alpha > 0$ and $d \leqslant 4$ any Gibbs state (such as characterized by the DLR equations[26]) is invariant under $\underline{\text{Glob}}$ (in particular, $\langle \phi_x \rangle_\alpha = 0$) independently of e^2, v^2, and λ.

(ii) $\underline{\text{Glob}}$ is unbroken for $v^2 < v_c^2(\lambda)$ and any $\alpha \geqslant 0$, $e^2 \geqslant 0$, where $v_c^2(\lambda) > 0$ is the critical value for SSB in the pure spin model ($e^2 = 0$).

Theorem 3.2(i) is proven in Ref. 12 (for $d = 4$) and Ref. 13 (for $d \leqslant 4$) by a spin wave argument similar to the McBrian–Spencer version[11] of the Mermin–Wagner theorem. As in Lemma 2.1, it essentially relies on the fact that $(\Delta^{-2})_{xx}$ is infrared-divergent for $d \leqslant 4$. Theorem 3.2(ii) is proven by the analogue correlation inequality as in the proof of Corollary 2.3.

Looking at Fig. 5a, we realize that for *large* λ, Theorem 3.2(ii) has the following consequences: Either $\langle \phi \rangle_{\text{Landau}}$ is not only zero in region I, but also in all of region II, and hence not a suitable order parameter for the Higgs/confinement phase; or $\langle \phi \rangle_{\text{Landau}}$ is only zero in region I and the "confinement subregion" (say, corresponding to IIb in Fig. 5b) and indeed nonzero in the "Higgs subregion" (i.e., for v^2 large enough), as predicted by perturbation theory. In this case, however, we would have to face the fact that the order parameter in the gauge-fixed model would indicate a phase transition where there is none in terms of local gauge-invariant quantities [Theorem 3.1(i)]. A similar situation has indeed been proven to occur in the $\mathbb{Z}_2$ Higgs model.[27] Whether a phase transition indicated by such a quantity would be of any physical relevance remains unclear.

Finally we turn to the symmetry Grib, for which even less is known so far.

Theorem 3.3. Let $d \geqslant 2$ and $\lambda > 0$ in the compact model.

(i) Gib is unbroken independently of λ, v^2, and e^2 if α is large enough.[12]

(ii) To any $\lambda > 0$ there exist constants $e_0^2 > 0$ and v_0^2 (v_0^2 may be negative) such that Grib is unbroken for $e^2 > e_0^2$ and $v^2 < v_0^2$, but independently of α.

We remark that by monotonicity properties in α (again due to correlation inequalities of Ref. 21) it is enough to prove Theorem 3.3(ii) for $\alpha = 0$. This has been done for the fixed-length model in Ref. 13.

The general case can be handled using similar bounds as in Appendix B, and will be published elsewhere.

We conclude this section with conjectural remarks on what might happen in Landau gauge (see Fig. 6). Based on Theorem 3.2(ii), we believe that

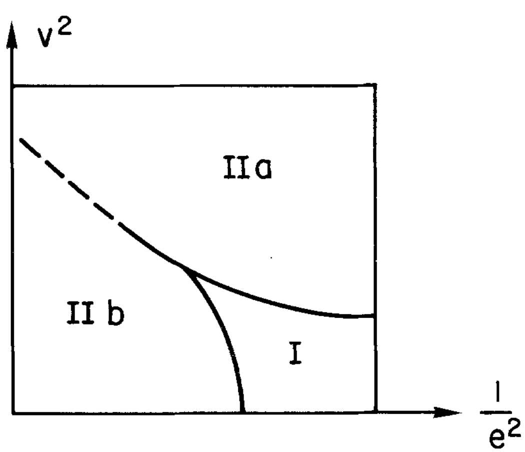

Fig. 6. Conjectured phases according to the symmetries Grib and Glob in Landau gauge of the compact model. We conjecture Grib and Glob to be unbroken in IIb, Grib and Glob to be broken in IIa, and Grib to be broken but Glob unbroken in I.

<u>Glob</u> is unbroken in the "Coulomb" region I and in some "confinement subregion" IIb, which in the case of small λ might coincide with region IIb of Fig. 5b. Motivated by the results of Szlachányi,[26] we suggest that <u>Glob</u> can be broken in the Higgs subregion IIa even for large values of λ, indicating a "fake phase transition," but confirming otherwise the standard picture of the Higgs mechanism.

According to Theorem 3.3(ii), <u>Grib</u> is at least unbroken in the confinement subregion IIb. If <u>Glob</u> is broken in the Higgs subregion IIa, then so is <u>Grib</u> $\supset$ <u>Glob</u> (besides, by a correlation inequality $\langle\phi(x)\,\phi(y)\rangle \geqslant |\langle\phi(x)\rangle|^2$, hence $\langle\phi_x\rangle_{\text{Landau}} \neq 0$ implies $\langle\phi(x)\,\phi(y)\rangle_{\text{Landau}} \neq 0$). Finally, we speculate that <u>Grib</u> might also be broken in the "Coulomb" region I, such that $\langle\phi(x)\,\phi(y)\rangle_{\text{Landau}}$ (for $x \neq y$) could be an order parameter for a "deconfinement" transition between IIb and I.

4. PROOFS

4.1. Proof of Theorem 2.2(i)

To show that the photon propagator is not summable, we use the sufficient condition of Ref. 25 (see also Refs. 2 and 3). We prove that for arbitrary real-valued one-forms j with finite support and $d^*j = 0$ (i.e., conserved currents) the following inequalities hold:

$$(j, \Delta^{-1}j) \geqslant \langle (e^{-1}A, j)^2 \rangle \geqslant \delta(j, \Delta^{-1}j) \tag{4.1}$$

with $\delta > 0$ provided $e^2 > \tilde{e}^2(v^2)$ for a function $\tilde{e}^2(\cdot)$ to be defined below.

For the upper bound in (4.1) we put $v^2 = 0$ in the action. By a correlation inequality,[21] this increases the expectation of $(A, j)^2$. But now the A integral is purely Gaussian and yields $(j, \Delta^{-1}j)$. The lower bound follows from the more general inequality

$$\langle \exp(e^{-1}A, j) \rangle \geqslant \exp[\tfrac{1}{2}\delta(j, \Delta^{-1}j)] \tag{4.2}$$

after rescaling $j \to sj$ and differentiating twice at $s = 0$ (there are no first-order terms in s). To prove (4.2), we first perform a change of variables $A = A' + e\Delta^{-1}j$ in the unnormalized expectation. This gives

$$\langle \exp(e^{-1}A, j) \rangle = \exp[\tfrac{1}{2}(j, \Delta^{-1}j)]\; E_C(h(j))/E_C(0) \tag{4.3}$$

with $h(j) = e\Delta^{-1}j$ and

$$E_C(h) = \int d\mu_C(A)\, D\varphi \exp\left[v^2 \sum_{\langle xy \rangle} \cos(\varphi_x - \varphi_y + A_{xy} + h_{xy}) \right]$$

Here $d\mu_C(A)$ is the Gaussian measure with covariance

$$C = e^2[d^*d + (1/\alpha)\, dd^*]^{-1}$$

Now we use a correlation inequality very similar to those of Ref. 21, which is proven in Ref. 13.

Lemma 4.1. $E_C(h)/E_C(0)$ is monotone increasing in C.
On the unit lattice we have $\Delta \leqslant 4d$ and hence

$$E_C(h)/E_C(0) \geqslant E_{\tilde{C}}(h)/E_{\tilde{C}}(0) \tag{4.4}$$

with

$$\tilde{C} := (e^2/4d)(d^*\Delta^{-1}d + \alpha d\Delta^{-1}d^*) \leqslant C \tag{4.5}$$

We are free to choose $\alpha = 1$ here (i.e., Feynman gauge), since we are considering a gauge-invariant expectation value. This gives $\tilde{C} = e^2/4d$ and hence

$$E_{\tilde{C}}(h) = \int D\varphi \left[\exp \sum_{\langle xy \rangle} s(\varphi_x - \varphi_y + h_{xy}) \right]$$

with

$$\exp[s(\theta)] = \left(\frac{e^2\pi}{2d} \right)^{1/2} \int_{-\infty}^{+\infty} dA \exp\left(\frac{2d}{e^2} A^2 \right) \exp[v^2 \cos(A + \theta)]$$

We see that $E_{\tilde{C}}(h)/E_{\tilde{C}}(0)$ is a disorder parameter in a plane rotator model with nearest neighbor couplings described by the action $s(\varphi_x - \varphi_y)$. This action interpolates between the Villain action with temperature $e^2/4d$ ($v^2 = \infty$) and the usual action of the standard XY model with temperature v^{-2} ($e^2 = 0$). If e^2 is large enough or v^2 is small enough, the plane rotator is in its high-temperature phase. By Fourier transformation we get an order parameter in a gas of closed loops ("defects") at low temperature. We write

$$e^{v^2 \cos\theta} = \sum_{n \in \mathbb{Z}} e^{-in\theta} I_n(v^2)$$

where I_n are the modified Bessel functions. Executing the Gaussian A integral, we obtain

$$E_{\tilde{C}}(h) = \int D\varphi \sum_{n_1 : \Lambda_1 \to \mathbb{Z}} \exp[i(h - d\varphi, n_1)]$$
$$\times \prod_{xy} \{\exp[-(e^2/8d)\, n_1(xy)^2]\, I_{n_1(xy)}(v^2)$$

where the sum is over all integer-valued one-forms n_1.

The φ integration now gives the constraint $d^*n_1 = 0$ and we arrive at

$$E_{\tilde{C}}(h) = \sum_{d^*n_1=0} \exp[i(n_1, h)] \prod_{\langle xy\rangle} \{\exp[-(e^2/8d)\, n_1(xy)^2]\}\, I_{n_1(xy)}(v^2) \tag{4.6}$$

Now we can use the methods of Guth[20] [who had arrived at this model with $v^2 = \infty$, starting from a Wilson loop expectation in the pure $U(1)$ gauge model] in the version of Seiler[22] to show (see Appendix A) the following result:

Lemma 4.2. To every $\delta < 1$ there is a constant $M > 0$ such that for all e^2 and v^2 with

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-(e^2n^2/8d)} I_n(v^2)/I_0(v^2) < e^{-M} \tag{4.7a}$$

we have

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} |\ln E_{\tilde{C}}(h) - \ln E_{\tilde{C}}(0)| < \frac{1-\delta}{2e^2}(dh, dh)$$

Combining (4.3), (4.4), and Lemma 4.2, we get the desired result:

$$\langle \exp(e^{-1}A, j)\rangle \geqslant \exp[\tfrac{1}{2}\delta(j, \Delta^{-1}j)]$$

provided $e^2 > \tilde{e}^2(v^2, M)$, where $\tilde{e}^2(v^2, M)$ is the solution of

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{-e^2n^2/8d} I_n(v^2)/I_0(v^2) = e^{-M} \tag{4.7b}$$

Next we prove $\langle\phi\rangle_{\text{Landau}} = 0$ by a similar technique. First we note that in any α-gauge, $\langle \phi_x\rangle_\alpha$ increases if we replace the covariance C in the Gaussian A-measure by $\tilde{C}$ given in (4.5) (due to correlation inequalities of Ref. 21). In the fixed-length model we may write $\phi_x = ve^{i(\varphi, q)}$ with $q(\cdot) = \delta_x(\cdot)$ and after the same Fourier transformation as before we arrive at

$$\langle \phi_x\rangle_{\text{Landau}} \leqslant vZ(q)/Z(0) \tag{4.8}$$

with

$$Z(q) = \sum_{n_1} \exp -\frac{e^2}{8d}(n_1 d^*\Delta^{-1}\, dn_1)$$
$$\times \prod_{\langle xy\rangle} I_{n_1(xy)}(v^2) \int D\varphi \exp[-i(d\varphi, n_1)] \exp[i(\varphi, q)]$$

where we have put $\alpha=0$ in $\tilde{C}$. Now the φ integration gives the constraint $d^*n_1=q$, yielding

$$Z(q)=\exp\frac{e^2}{8d}(q,\Delta^{-1}q)\,\tilde{Z}(q) \tag{4.9}$$

with

$$\tilde{Z}(q)=\sum_{d^*n_1=q}\prod_{\langle xy\rangle}\left\{\left[\exp\left(-\frac{e^2}{8d}n_{xy}^2\right)\right]I_{n_1(xy)}(v^2)\right\}$$

where we have used $d^*\Delta^{-1}d=\mathbb{1}-d\Delta^{-1}d^*$.

Hence $\tilde{Z}(q)/\tilde{Z}(0)$ is a disorder parameter in the same gas of closed loops as before. By inverse Fourier transformation we come back to the expectation of the order parameter $e^{i(\varphi,q)}\equiv e^{i\varphi_x}$ in the above plane rotator model:

$$\tilde{Z}(q)=\int D\varphi[\exp i(\varphi,q)]\left[\exp\sum_{\langle xy\rangle}s(\varphi_x-\varphi_y)\right]$$

Defining $\rho(\theta)$ according to

$$e^{s(\theta)}=I_0(v^2)[1+\rho(\theta)]$$

one can do a standard high-temperature cluster expansion.[22,28] If $\|s\|_\infty<e^{-M}$ with M large enough, this expansion converges absolutely and uniformly in Λ. Thus, the thermodynamic limit exists and is independent of boundary conditions. Using free boundary conditions, we conclude that

$$\lim_{\Lambda\to\mathbb{Z}^d}\tilde{Z}(q)/\tilde{Z}(0)=0 \tag{4.10}$$

whenever $\sum_x q(x)\neq 0$. Combining (4.8)–(4.10) with the identity

$$\rho(\theta)=\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}e^{in\theta}\,e^{-e^2n^2/8d}\,I_n(v^2)/I_0(v^2)$$

and the fact that in $d\geqslant 3$, $(q,\Delta^{-1}q)<\infty$, we have proven that

$$\lim_{\Lambda\to\mathbb{Z}^d}\langle\phi_x\rangle_{\text{Landau}}=0$$

provided $e^2>\tilde{e}(v^2,M)$ with M large enough, where $\tilde{e}^2(\cdot)$ is defined in (4.7).

Finally, we discuss the shape of $\tilde{e}^2(\cdot,M)$. Since $I_n(v^2)/I_0(v^2)$ is strictly monotone increasing, it is clear that the same holds for $\tilde{e}^2(\cdot,M)$. The boundedness of $\tilde{e}^2(\cdot, M)$ following since $I_n(v^2)/I_0(v^2) \to 1$ as $v^2 \to \infty$ and $\tilde{e}^2(v_0^2, M) = 0$, where v_0^2 is the solution of

$$\sum_{n \neq 0} I_n(v_0^2)/I_0(v_0^2) = e^{-M} \quad \blacksquare$$

4.2. Proof of Theorem 2.4(i) for $v^2 < 0$

In this section we combine the methods of Ref. 2 with correlation inequalities and random walk expansions to show that the photon propagator is not summable for arbitrary $e^2 > 0$ and $\lambda > 0$, provided $v^2 < 0$.

We first note that it is enough (see Section 4.1) to show that for arbitrary real-valued 1-forms j with $d^*j = 0$

$$\langle (A, j)^2 \rangle_\Lambda \geqslant \bar{e}^2(j, \Delta^{-1} j)_\Lambda \tag{4.11}$$

with a constant $\bar{e}^2 > 0$ independent of Λ and j. Due to the correlation inequalities of Ref. 20, the expectation value of $(A, j)^2$ falls if the potential $V(\phi) = \lambda(|\phi|^2 - v^2)^2$ is replaced by

$$V'(\phi) = \tfrac{1}{2} m^2 |\phi|^2 \tag{4.12}$$

with $m^2 = -2\lambda v^2$. Therefore it is enough to show (4.11) for the model with potential V'.

For this model, however, the bound (4.11) is an immediate consequence of the results of Ref. 2 and the random walk expansion for $\det(\Delta_A + m^2)$, where Δ_A is the covariant Laplacian: Let

$$S_\Lambda^{\text{eff}}(A) = -\log Z_\Lambda(A)$$

$$Z_\Lambda(A) = \int D_\Lambda \phi \exp\left[-\frac{m^2}{2} \sum_x |\phi_x|^2 - \frac{1}{2} \sum_{\langle xy \rangle} |(D_A \phi)_{xy}|^2 \right]$$

Then, by standard methods from the theory of random walks (see, e.g., Ref. 29)

$$S_\Lambda^{\text{eff}}(A) = \text{const} - \sum_{\substack{\omega \\ |\omega| \neq 0}} \frac{1}{|\omega|} \left(\frac{1}{m^2 + 2d} \right)^{|\omega|} \cos\left(\sum_{b \in \omega} A_b \right) \tag{4.13}$$

where the constant is independent of A. The sum goes over closed random walks in Λ, i.e., finite sequences

$$\omega = (\langle x_1 y_1 \rangle, \ldots, \langle x_n y_n \rangle)$$

of n.n. pairs in Λ_1 such that $y_i = x_{i+1}$ and $x_1 = y_n$. Here $|\omega|$ is the number of steps n of ω and $\sum_{b \in \omega} A_b$ is a symbolic notation for $\sum_{i=1}^{n} A_{x_i y_i}$. The sum (4.13) is absolutely convergent and even obeys the stronger convergence condition

$$\sum_{\omega : b \in \omega} e^{\varepsilon |\omega|} \frac{1}{|\omega|} \left(\frac{1}{m^2 + 2d} \right)^{|\omega|} < \infty \tag{4.14}$$

for a suitably chosen constant $\varepsilon > 0$ which is independent of Λ and the particular bond $b \in \Lambda_1$. Due to (4.13) and (4.14), we can use Theorem 2.1 from Ref. 2 to get (4.11) for the model with potential V'. ∎

4.3. Proof of Theorem 2.5(i). We consider observables f depending only on the electromagnetic field dA and apply the transformation of Balaban *et al.*[4]:

$$\langle f \rangle_\Lambda = Z_\Lambda(f)/Z_\Lambda(1)$$

with

$$Z_\Lambda(f) = \int d\mu_\Lambda(A)\, d\nu_\Lambda(R) \sum_{\substack{n_2 : \Lambda_2 \to 2\pi\mathbb{Z} \\ dn_2 = 0}} \exp\left(-\frac{1}{2e^2} \|dA + n_2\|^2 \right)$$
$$\times \left\{ \exp\left[\sum_{\langle xy \rangle} (R_x R_y - \tilde{v}^2) \cos A_{xy} \right] \right\} f(dA + n_2) \tag{4.15}$$

The normalized *a priori* measures are given by

$$d\mu_\Lambda(A) = \prod_{\langle xy \rangle \in \Lambda_1} N_1(\tilde{v}^2)^{-1} [\exp(\tilde{v}^2 \cos A_{xy}) \chi_{[-\pi,\pi)}(A_{xy})\, dA_{xy}$$

$$d\nu_\Lambda(R) = \prod_{x \in \Lambda_0} N_2(\lambda, \tilde{v}^2)^{-1} \{\exp[-\lambda(R_x^2 - \tilde{v}^2)^2]\} \chi_{[0,\infty)}(R_x)\, R_x\, dR_x$$

with suitable normalization constants N_1, N_2 and $\tilde{v}^2 = v^2 - d/2\lambda$. Here $\chi_{[a,b)}$ is the characteristic function of $[a, b)$. The sum in (4.15) is over all 2-forms with values in $2\pi\mathbb{Z}$ which obey the "Bianchi identity" $dn_2 = 0$. Relation (4.15) is obtained by first performing a change of variables $A_{xy} + \varphi_y - \varphi_x = A'_{xy}$ and then splitting the noncompact integral over A' into a compact one (i.e., from $-\pi$ to π) and a sum over 1-forms with values in $2\pi\mathbb{Z}$,

$$\int_{-\infty}^{\infty} dA_b\, \psi(A_b) = \sum_{n_1(b) \in 2\pi\mathbb{Z}} \int_{-\pi}^{\pi} dA_b\, \psi(A_b + n_1)$$

for any bond b and integrable function ψ. Then one rearranges the sum over n_1 according to[5]

$$\sum_{n_1} = \sum_{n_2 : dn_2 = 0} \; \sum_{n_1 : dn_1 = n_2}$$

Now, for fixed n_2, the sum over gauge-equivalent configurations n_1 with $dn_1 = n_2$ combined with the angular integration over φ factorizes and cancels out (see Ref. 4 for details). Hence, the gauge-fixing term disappears and we arrive at (4.15) (one might call this the "unitary gauge" for the noncompact model).

To derive a polymer expansion for the unnormalized expectation $Z_\Lambda(f)$, we put $\rho_p = \exp(S_p + \tilde{S}_p) - 1$ with

$$\begin{aligned} S_p &= -\frac{1}{2e^2} \left[(dA)(p)^2 + 2n_2(p)\, dA(p) \right] \\ \tilde{S}_p &= \frac{1}{2(d-1)} \sum_{\langle xy \rangle \in \partial p} (R_x R_y - \tilde{v}^2) \cos A_{xy} \end{aligned} \tag{4.16}$$

This gives

$$Z_\Lambda(f) = \sum_{B \subset \Lambda_2} \; \sum_{n_2 : dn_2 = 0} k_\Lambda(B, n_2, f)$$

with

$$\begin{aligned} k_\Lambda(B, n_2, f) = &\left[\exp\left(-\frac{1}{2e^2} \|n_2\|^2 \right) \right] \\ &\times \int d\mu_\Lambda(A)\, d\nu_\Lambda(R)\, f(dA + n_2) \prod_{p \in B} \rho_p \end{aligned}$$

We now gather all B and n_2 such that $\Gamma := B \cup \operatorname{supp} n_2$ is fixed:

$$Z_\Lambda(f) = \sum_{\Gamma \subset \Lambda_2} \; \sum_{\substack{B, n_2 : dn_2 = 0 \\ B \cup \operatorname{supp} n_2 = \Gamma}} k_\Lambda(B, n_2, f) \tag{4.17}$$

Next we decompose Γ into polymers: *Ordinary polymers* are non-empty, connected subsets $\gamma \subset \Lambda_2$ and f-polymers are either empty or a union of ordinary polymers, each of which is connected to the "space-time"

[5] We always work with rectangular open boxes such that the νth cohomology is trivial for $0 \leqslant \nu \leqslant d-1$.

support of f. Here we call plaquettes connected if they join a common corner or if they are faces of a common cube. The "space-time"support of f is the minimal set $P \subset \Lambda_2$ such that f does not depend on $F(p) \equiv (dA)(p)$ for $p \notin P$. Dividing each Γ in (4.17) into a maximal f-polymer $\Gamma_f \subset \Gamma$ and the connectivity components $\gamma \in \text{conn}(\Gamma \backslash \Gamma_f)$ of $\Gamma \backslash \Gamma_f$, we get

$$Z_\Lambda(f) = \sum_{\Gamma \subset \Lambda_2} z_\Lambda(\Gamma_f, f) \prod_{\gamma \in \text{conn}(\Gamma \backslash \Gamma_f)} z_\Lambda(\gamma) \tag{4.18}$$

with

$$z_\Lambda(\gamma, f) = \sum_{B} \sum_{\substack{dn_2 = 0 \\ B \cup \text{supp}\, n_2 = \gamma}} k_\Lambda(B, n_2, f)$$

and $z_\Lambda(\gamma) \equiv z_\Lambda(\gamma, 1)$. Using the usual techniques of Mayer expansions for polymer systems,[22,28] one now obtains

$$\langle f \rangle_\Lambda = \sum_{n=0}^{\infty} \sum_{\gamma_0}{}^{f} \sum_{\gamma_1,\ldots,\gamma_n} \frac{\phi_c(\gamma_0,\ldots,\gamma_n)}{n!} z_\Lambda(\gamma_0, f) \prod_{i=1}^{n} z_\Lambda(\gamma_i) \tag{4.19}$$

where the second sum is over f-polymers and the third one over ordinary polymers. $\phi_c(\gamma_0,\ldots,\gamma_n)$ is a combinatoric coefficient, which is zero if $\gamma_0 \cup \cdots \cup \gamma_n$ is not an f-polymer. In Appendix B we prove absolute convergence of (4.19) for polynomially bounded observables f, provided $\lambda > \lambda_1$, $e^2 < e_1^2$, $ev > e_1 v_1$ with suitable positive constants e_1, v_1, λ_1 which only depend on the space-time dimension d.

The exponential decay of the photon propgator now follows by standard arguments. First note that for observables $f(dA)$ with $f(-dA) = -f(dA)$ we immediately have $z_\Lambda(\gamma, f) = 0$ for all f polymers γ. Let now $f_1 = (dA)(p_1)$, $f_2 = (dA)(p_2)$, and $f = f_1 f_2$. Then

$$z_\Lambda(\gamma, f) = z_\Lambda(\gamma_1, f_1)\, z_\Lambda(\gamma_2, f_2) = 0$$

whenever γ contains no connected component that is connected to p_1 *and* to p_2 (γ_i is the union of those components of γ that are connected to p_i). Therefore the cluster expansion for the photon propagator contains only f-polymers, whose size is at least as large as the distance between p_1 and p_2. Since the activity of a polymer falls off exponentially with its size, this proves the exponential decay of $\langle dA(p_1)\, dA(p_2) \rangle$. ∎

APPENDIX A

In this Appendix we prove Lemma 4.2, using the methods of Seiler[22] for his proof of Guth's theorem.[20] We start with a standard low-temperature cluster expansion for $\log[E_{\tilde{C},\Lambda}(h)]$ with $E_{\tilde{C},\Lambda}(h)$ given in the form (4.6). To this end we first sum over all n_1 with a given support Γ and $d^*n_1=0$ and then over all sets $\Gamma\subset\Lambda_1$. This gives

$$E_{\tilde{C},\Lambda}(h)=\left[1+\sum_{\varnothing\neq\Gamma\subset\Lambda_1}\prod_{\gamma\in\operatorname{conn}\Gamma}\xi(\gamma,h)\right]I_0(v^2)$$

where $\operatorname{conn}(\Gamma)$ are the usual conectivity components (i.e., two bonds are connected if they have a common endpoint[6] and the activities ξ_Λ are given by

$$\xi_\Lambda(\gamma,h)=\sum_{\substack{n_1:\operatorname{supp}n_1=\gamma\\ d^*n_1=0\text{ on }\Lambda_0}} e^{i(n_1,h)}\prod_{\langle xy\rangle\in\gamma}\hat{\rho}(n_1(xy))$$

with

$$\hat{\rho}(n)=e^{-(e^2/8d)n^2}I_n(v^2)/I_0(v^2) \tag{A.1}$$

Under the condition of Lemma 4.2, we have

$$|\xi_\Lambda(\gamma,h)|<e^{-M|\gamma|}$$

and hence for M large enough, the Mayer expansion for $\log E_{\tilde{C},\Lambda}(h)$ converges. Doing the same thing for $h=0$, we obtain

$$\begin{aligned}&\log E_{\tilde{C},\Lambda}(h)-\log E_{\tilde{C},\Lambda}(0)\\&\quad=\sum_{n=1}^{\infty}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ \gamma_i\subset\Lambda}}\frac{\psi_C(\gamma_1,\dots,\gamma_n)}{n!}\left[\prod_{i=1}^{n}\xi_\Lambda(\gamma_i,h)-\prod_{i=1}^{n}\xi_\Lambda(\gamma_i,0)\right]\end{aligned}$$

Now to each n_1 with $d^*n_1=0$ there exists an integer-valued two-form $n_2(n_1)$ with $d^*n_2(n_1)=n_1$ such that the support of $n_2(n_1)$ is contained within the smallest rectangular box containing $\operatorname{supp}n_1$.[2] Writing

$$e^{i(n_1,h)}=e^{i(n_2(n_1),dh)}$$

and using the estimates of Seiler,[22] we conclude that to every $\varepsilon>0$ there exists a constant $K(\varepsilon)$ such that

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}|\log E_{\tilde{C},\Lambda}(h)-\log E_{\tilde{C},\Lambda}(0)|\leqslant(dh,dh)\,K(\varepsilon)\,S(\varepsilon;e^2,v^2) \tag{A.2}$$

with

$$S(\varepsilon;e^2,v^2)=\sum_{n=1}^{\infty}\sum_{\substack{\gamma_1,\dots,\gamma_n\\ b\in\gamma_1\cup\cdots\cup\gamma_n}}\left|\frac{\psi_c(\gamma_1,\dots,\gamma_n)}{n!}\prod_{i=1}^{n}e^{\varepsilon|\gamma_i|}\xi(\gamma_i,0)\right|$$

[6] We use the fact that for an integer-valued 1-form $n_1=j+k$ with $\operatorname{supp}j$ and $\operatorname{supp}k$ disconnected $d^*n_1=0$ implies $d^*j=0$ and $d^*k=0$.

Here

$$\xi(\gamma, 0) = \lim_{\Lambda \to \mathbb{Z}^d} \xi_\Lambda(\gamma, 0)$$

and the sum is over all n-tuples of polymers of the infinite lattice such that an arbitrary but fixed bond b is contained in at least one of them. By standard estimates for Mayer expansions[22,28] there exist constants $m_0 > 0$ and $K_0 > 0$ such that for all $m \geqslant m_0$

$$S(\varepsilon; e^2, v^2) \leqslant K_0 e^{-m} \tag{A.3}$$

provided $|\xi(\gamma, 0)| e^{\varepsilon|\gamma|} \leqslant e^{-m|\gamma|}$. This is achieved by choosing e^2 and v^2 such that the condition (4.7a) of Lemma 4.2 holds, i.e.,

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{\rho}(n) \leqslant e^{-m-\varepsilon}$$

where $\hat{\rho}$ is given in (A.1). The desired result follows from the next lemma.

Lemma A.1. To any constant $K_1 > 0$ there exists $M_1(K_1) > 0$ such that for all e and v the condition (4.7) with $M = M_1(K_1)$, i.e.,

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{\rho}(n) \leqslant e^{-M_1(K_1)}$$

implies

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{\rho}(n) \leqslant K_1/e^2$$

Proof. Choose e_1^2 such that

$$\sum_{n \neq 0} \exp(-e^2 n^2/8d) \leqslant K_1/e^2$$

for all $e^2 \geqslant e_1^2$ and put $M_1(K_1) := \ln(e_1^2/K_1)$. Lemma A.1 follows since $|I_n(v^2)/I_0(v^2)| \leqslant 1$. ∎

Given $\varepsilon > 0$ and $K_1 = (1-\delta)/2K_0 K(\varepsilon)$, whoose now in (4.7)

$$M := \max\{m_0 + \varepsilon, M_1(K_1) + \varepsilon\}$$

The condition (4.7) then implies $|\xi(\gamma, 0)| e^{\varepsilon|\gamma|} \leqslant e^{-m|\gamma|}$ with

$$e^{-m} \leqslant \min\{e^{-m_0}, (1-\delta)/2K_0 K(\varepsilon) e^2\}$$

by Lemma A.1. Using the bounds (A.2) and (A.3), we conclude

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} |\log E_{\tilde{C},\Lambda}(h) - \log E_{\tilde{C},\Lambda}(0)| \leqslant \frac{1-\delta}{2e^2}(dh, dh) \quad \blacksquare$$

APPENDIX B. CONVERGENCE OF THE CLUSTER EXPANSION (4.19)

In this Appendix we combine the methods of Refs. 7 and 23 to prove the following result.

Lemma B.1. $\forall \varepsilon > 0$ there are finite positive constants e_1, λ_1, and v_1 depending only on the dimension d and on ε, such that for $ve \geqslant v_1 e_1$, $e^2 \leqslant e_1^2$, $\lambda \geqslant \lambda_1$,

$$|z_\Lambda(\gamma, f)| \leqslant K_{e_1}(f)\varepsilon^{|\gamma|} \tag{B.1}$$

with

$$K_{e_1}(f) := \sup_{A,n_2} \left\{ \exp\left[-\frac{1}{8e_1^2}(n_2, n_2) \right] \right\} |f(dA + n_2)|$$

which implies absolute convergence of the cluster expansion (4.19) for a large class of observables (including those that are polynomially bounded), provided $v^2 e^2$ and λ are large enough and e^2 is small enough.

To prove Lemma B.1, we recall the definition of $z_\Lambda(\gamma, f)$,

$$z_\Lambda(\gamma, f) = \sum_{\substack{B, n_2 : dn_2 = 0 \\ \operatorname{supp} n_2 \cup B = \gamma}} k_\Lambda(B, n_2, f) \tag{B.2}$$

with

$$k_\Lambda(B, n_2, f) = \left\{ \exp\left[-\frac{1}{2e^2}(n_2, n_2) \right] \right\} \left\langle f(dA + n_2) \prod_{p \in B} \rho_p \right\rangle_\Lambda^0$$

where $\langle \cdot \rangle_\Lambda^0$ denotes expectations with respect to the measure $d\mu_\Lambda(A)\, d\nu_\Lambda(R)$ and ρ_p is of the form

$$\rho_p = \exp[S_p(A, n_2) + \tilde{S}_p(A, R)] - 1$$

Let $\|\cdot\|_p$ denote the p-norm with respect to $d\mu_\Lambda(A)\, d\nu_\Lambda(R)$ and let $f_{n_2}(dA) := f(dA + n_2)$. Using Hölder's inequality, one can bound {note that $A_{xy} \in [-\pi, \pi)$}

$$|k_\Lambda(B, n_2, f)| \leqslant \left\{ \exp\left[-\frac{1}{2e^2}(n_2, n_2) \right] \|f_{n2}\|_\infty \prod_{i=1}^{k} \left\| \prod_{p \in B_i} \rho_p \right\|_k \right.$$

if $B_1,\dots, B_k$ are disjoint sets with $B = B_1 \cup \cdots \cup B_k$. Since there are not more than $r = 2d(d-1)$ plaquettes containing a given site, it is possible to decompose Λ_2 into r disjoint sets $\Lambda_2^1,\dots, \Lambda_2^r$ such that Λ_2^i is completely disconnected (i.e., p and $p' \in \Lambda_2^i$ have no common corner unless $p = p'$). But then ρ_p and $\rho_{p'}$ have no common variable if p, $p' \in \Lambda_2^i$ and $p \neq p'$. Therefore, by choosing $k = r$ and $B_i = B \cap \Lambda_2^i$, we get

$$|k_\Lambda(B, n_2, f)| \leqslant \left\{ \exp\left[-\frac{3}{8e^2}(n_2, n_2) \right] \right\} K_e(f) \prod_{p \in B} \|\rho_p\|_r \tag{B.3}$$

Next we bound

$$\begin{aligned} \|\rho_p\|_r &= \|e^{S_p + \tilde{S}_p} - 1\|_r \\ &\leqslant \|e^{\tilde{S}_p} - 1\|_r + \|e^{\tilde{S}_p}\|_{2r} \|e^{S_p} - 1\|_{2r} \\ &= \left\| \int_0^1 \tilde{S}_p e^{t\tilde{S}_p}\, dt \right\|_r + \|e^{\tilde{S}_p}\|_{2r} \|e^{S_p} - 1\|_{2r} \\ &\leqslant \|e^{|\tilde{S}_p|}\|_{2r} (\|\tilde{S}_p\|_{2r} + \|e^{S_p} - 1\|_{2r}) \end{aligned} \tag{B.4}$$

where we have used Hölder's inequality again. Notice that

$$\begin{aligned} |\tilde{S}_p| &\leqslant \frac{1}{2(d-1)} \prod_{\langle xy \rangle \in \partial p} |R_x R_y - \tilde{v}^2| \\ &\leqslant \frac{1}{2(d-1)} \sum_{\langle xy \rangle \in \partial p} (\tilde{v}\,|R_x - \tilde{v}| + \tilde{v}\,|R_y - \tilde{v}| + |R_y - \tilde{v}|\,|R_y - \tilde{v}|) \\ &\leqslant \frac{1}{2(d-1)} \sum_{x \in C(p)} (2\tilde{v}\,|R_x - \tilde{v}| + |R_x - \tilde{v}|^2) \\ &\leqslant \frac{1}{2(d-1)} \sum_{x \in C(p)} 3\,|R_x^2 - \tilde{v}^2| \end{aligned}$$

where we have used that

$$|R_x - v|\,|R_y - v| \leqslant (|R_x - v|^2 + |R_y - v|^2)/2$$

in the third inequality and the fact that $\tilde{v}$ and $|\tilde{v} - R_x|$ are bounded by $\tilde{v} + R_x$ in the last one. $C(p)$ denotes the corners of p.

Hence we can bound

$$\|\exp|\tilde{S}_p|\,\|_{2r} \leqslant \prod_{x \in C(p)} \|\exp(K|R_x - \tilde{v}^2|)\|_1^{1/2r}$$

where $K = (2r) \times 3/2(d-1)$.

By elementary estimates one gets

$$\|R_x^2 - \tilde{v}^2\|_{2r} \leqslant \frac{1}{\sqrt{\lambda}} \left[\frac{2}{\pi} \int_{-\infty}^{+\infty} x^{2\gamma} \exp(-x^2)\, dx \right]^{1/2r}$$

$$\|\exp(K|R_x^2 - \tilde{v}^2|)\|_1 \leqslant \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(K|x|/\lambda) \exp(-x^2)\, dx$$

$$\leqslant \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(K|x|) \exp(-x^2)$$

where in the last step we have assumed $\lambda \geqslant 1$. We thus have proven the existence of a constant K_1 depending only on the space-time dimension d, such that for all $\lambda \geqslant 1$

$$\|\tilde{S}_p\|_{2r} \leqslant K_1 \lambda^{-1/2r}, \qquad \|e^{|\tilde{S}_p|}\|_{2r} \leqslant K_1 \tag{B.5}$$

To bound $\|e^{S_p} - 1\|_{2r}$, we use the results of Ref. 7, Section 5, which imply the existence of a constant K_2 depending on d, such that for $e^2\tilde{v}^2 \geqslant 1$

$$\|e^{Sp} - 1\|_{2r} \leqslant K_2/e^2\tilde{v}^2 \tag{B.6}$$

if $n_2(p) = 0$ and

$$\|e^{Sp} - 1\|_{2r} \leqslant K_2 \exp \frac{n_2(p)^2}{4e^2} \tag{B.7}$$

if $n_2(p) \neq 0$.

Now (B.2)–(B.7) clearly imply the desired inequality (B.1), provided $\tilde{v}^2 e^2 \geqslant \tilde{v}_1^2(\varepsilon)\, e_1^2(\varepsilon)$, $e^2 \leqslant e_1^2(\varepsilon)$, $\lambda \leqslant \lambda_1(\varepsilon)$, with suitable constants $\tilde{v}_1^2(\varepsilon)$, $e_1^2(\varepsilon)$, $\lambda_1(\varepsilon)$. To complete the proof of Lemma B.1, we define $v_1^2(\varepsilon) := \tilde{v}_1^2(\varepsilon) + (d/2)\, \lambda_1(\varepsilon)$ and note that with this definition

$$v^2 e^2 \geqslant v_1^2 e_1^2, \qquad \lambda \geqslant \lambda_1, \qquad e^2 \leqslant e_1^2$$

imply

$$\tilde{v}^2 e^2 \geqslant \tilde{v}_1^2 e_1^2 \quad \blacksquare$$

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